With an understanding of input and output tolerances, we are now ready to define the idea of limit. The best example for that is the notion of a continuous function. Intuitively a continuous function is a function whose graph can be drawn without lifting your pencil. Consider graphing the function $f(x)$:

$$f(x) = \begin{cases} 
3x - 1 & x < 2 \\
3x + 1 & x \geq 2 
\end{cases}$$

We start at the left of the graph (small $x$) and move our pencil up and down as we move from left to right. If there are no jumps, then as the $x$ values get to a particular point, say $x=1$, the $y$ values will be very close to the height of the graph above 1 which is equal to $f(1) = 2$. How do we convince someone of that without just pointing to the graph? The key is to be able to be specific about what we mean by ‘very close’ to $f(1)=2$. What we mean is “as close as we want!” So let’s say we want to be within an output tolerance of .1 to $f(1)=2$. Then the output range is [1.9, 2.1]. To get the input range I need to solve

$2-0.1=3x-1 \implies 3-0.1=3x \implies x = 1-0.1/3 = .9667.$

$2+0.1 = 3x+1 \implies 3+0.1 = 3x \implies x = 1+0.1/3 = 1.0333.$
Since both .9667 and 1.0333 are at a distance of .0333 from our input target value of 1, the input tolerance is 0.0333. Suppose you wanted an even stricter output tolerance of .01. Then we can just run the argument again and get an input tolerance of 0.01/3 = .0333. So no matter what output tolerance we set, we can find an input tolerance around the point x=1 that guarantees our output lies close to the value f(1) = 2. We write that as

$$\lim_{x \to 1} f(x) = 2$$

which is read as “the limit as x goes to 1 of f of x is equal to 2.

So what happens as x approaches 2? From the definition, f(2) = 7. So let’s take a very loose output tolerance around 7 of 3. Then the input range is [4,10]. To find the input range we solve

$$4 = 3x - 1 \quad \Rightarrow \quad 5 = 3x \quad \Rightarrow \quad x = 5/3 = 1.667.$$  
$$10 = 3x + 1 \quad \Rightarrow \quad 9 = 3x \quad \Rightarrow \quad x = 3$$

So the input range is [1.667, 3] tolerance is the minimum of |1.667-2| and |3-2| which is .333. Notice that in calculating the left hand edge of the input range x<2 so we use f(x) = 3x-1, and in calculating the right hand edge of the input range x>2 so we use f(x) = 3x+1.

But suppose we want to make sure that we’re closer to the output value f(2)=7. Say we take an output tolerance of 0.5. Then the output range is [6.5, 7.5]. If we try to solve for the left edge of the input range, we have

$$6.5 = 3x - 1 \quad \Rightarrow \quad 7.5 = 3x \quad \Rightarrow \quad x = 7.5/3 = 2.5.$$  

But if we’re looking for the left hand edge of the input range, we need x<2. So the answer x=2.5 doesn’t make any sense! But if you look at the graph, we see that no matter how close the values of x get to 2, as long as they are less than 2 the function doesn’t get any bigger than 5. So there is no way to get closer than .5 units away from our target value of 7, no matter how small we make the input tolerance. That’s because the output tolerance is less than the jump. Turning things around, the notion of input and output tolerances give us a way of talking about ‘jumps’ in a function, even if we can’t see the graph. In terms of limits, we say that the limit

$$\lim_{x \to 2} f(x) \quad \text{'does not exist.'}$$

Now we can translate what we mean when we say that a continuous function is a function whose graph does not have any jumps. Tweaking that just a little, we can say that for a continuous function, every input value x is NOT a jump. In terms of limits this means for any particular x value called x₀ it’s true that

$$\lim_{x \to x_0} f(x) = f(x_0)$$
So this seems like a lot, just to translate what it means to draw a graph without picking up your pencil. More than that, all of the basic functions in this course are continuous and almost all the ways of combining continuous functions (adding, subtracting, multiplying, composing) also lead to continuous functions. What’s left are two basic ways to get a discontinuous function (a function that is not continuous at at least one point): (i) the definition is presented in parts (like the function in the example), or (ii) there is a place where you divide by 0.

Because there are only a couple of way to make discontinuous functions, all the problems like “is this function continuous” tend to have a similar form. I think many students tend to memorize how these go without really understanding the limit or what things mean. That’s OK as far as it goes, but continuity is one of the topics that makes for good examples illustrating what it means when we take limits. Since we’ll be taking many different kinds of limits in this course, the more you can get a gut feeling for what it means, the easier it will be to understand these future topics.

To finish off the topics of continuity, we define the idea of a one sided limit. In the example of our simple function f(x) that has a jump at x=2, it’s easy to see that if we take input values x that approach to 2 from the left, then f(x) will get closer and closer to $3*2-1 = 5$. This is the left-hand ‘jumping off point’ for the jump in the graph. If x approaches 2 from the right, then the values of f(x) get closer and close to $3*2+1=7$. We write these **one sided limits** as

$$\lim_{x \to 2^-} f(x) = 5 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 7$$

interpreting x-$\to$2- as meaning “x approaches to from the negative side” or “x approaches to from below” and interpreting x-$\to$2+ to mean “x approaches to from the positive side” or “x approaches to from above.”

These limits are defined by using the same input/output tolerance idea, it’s just that the input range is now one-sided. For example, suppose we look at $\lim_{x \to 2^-} f(x) = 5$ using an output tolerance of .1. Then the output range is [4.9 5.1]. To get the input values that correspond to that range we’re restricting ourselves to x<$\to$2 so in this range $f(x) = 3x-1$. For the left end point of the input range we solve

$4.9=3x-1 \rightarrow 5.9=3x \rightarrow x=5.9/3=1.967$.

From the graph, we see that f(x)<5 for x<$\to$2. That means if x is in the one-sided input range of [1.967, 2) then the output falls inside the input range of [4.9 5.1]. It’s pretty easy to see that for any output tolerance, we can find an input tolerance such that if x<$\to$2 and x is closer to 2 than that input tolerance, then f(2) will be closer to 5 than the specified output tolerance. Notice that in specifying the range, I used a round parentheses for the upper end. This is a technicality that means that the value 2 is NOT included in the range, while the square bracket means that 1.967 IS included. This is needed because f(2) = 7, which lies outside our output range.
A similar argument applies to \( \lim_{x \to 2^+} f(x) = 7 \). If we take an output tolerance of .1 then the output range is [6.9, 7.1]. For all values \( x > 2 \), \( f(x) = 3x + 1 \) and we can solve

\[
7.1 = 3x + 1 \implies 6.1 = 3x \implies x = 6.1 / 3 = 2.033
\]

to find a one-sided input range of (2, 2.033]. If \( x \) is in that range then \( f(x) \) falls in the specified output range.