MAT1193 - 4f Graphs of first and second derivative

One of the most important applications of the derivative is that the value of the derivative at a given base point indicates whether the original function is increasing or decreasing near that base point. This is pretty obvious: if a function is increasing as you go from left to right near a base point, then the slope of the tangent line at that point will be positive. Conversely, if a function is decreasing near a base point, then the slope of the tangent line at that point will be negative. Notice that the value of the function can be negative, while the derivative is positive and vice versa.

Now we can use this fact to determine where a function is increasing or decreasing. For example, suppose we are given the function $f(x) = 3x^2 - 6x$. For what values of $x$ is the function increasing/decreasing? Calculating the derivative, we have $f'(x) = 6x - 6$. This derivative is positive when $x > 1$ and so the function $f(x)$ is increasing for $x > 1$. For $x < 1$ the function is decreasing. So the derivative can tell us important things about the graph of a function.

What is the relationship between the graph of a function $f(x)$ and the graph of its derivative $f'(x)$? Well, any values of $x$ where $f(x)$ is increasing, the value of $f'(x)$ is positive, so when $f(x)$ is increasing the graph of $f'(x)$ is above the x axis. Conversely, when $f(x)$ is decreasing, the graph of $f'(x)$ lies below the x axis. An easy way to see the relationship between the two graphs is to focus on the critical points. A critical point is a point where the derivative is equal to zero. (If there is a point where the derivative is not defined that is also a critical point, but we won't emphasize these types of critical points in this class.) For the original function, the critical points are points where the graph changes from increasing to decreasing or vice versa. For the derivative function, the critical points are points where the graph goes through the x axis.
The second derivative.

The second derivative is simply the derivative of the derivative. That is, it tells you about how the slope of the tangent line is changing as you move from left to right. What does that tell you about the original function? Let's consider an example of a function \( f(x) \), where we know that \( f(x_0) < 0 \), \( f'(x_0) > 0 \), \( f''(x_0) > 0 \). The fact that \( f(x_0) < 0 \) means that the graph of \( f \) at the point \( x_0 \) is below the \( x \)-axis. The fact that \( f'(x_0) > 0 \) means that the function is increasing. That means if we consider the point \( x_0 \) and two other points near \( x_0 \) but on either side, the value of the function will be negative for all three points but it will be more negative for the point on the left and less negative (more positive) for the point on the right (see the figure to the left). Now because the slope of the tangent line is positive at the point \( x_0 \) we expect it to be positive for the two nearby points as well. But the fact that the second derivative \( f''(x_0) > 0 \), means that the slope of the tangent line will be increasing as you move from left to right, so that the point to the left will have a tangent line with a slope that is less positive than at \( x_0 \), while the point to the right will have a tangent line with a slope that is more positive than at \( x_0 \) (see the figure to the right). That means that the graph is "curving upward", a property we call concave up.
Now let's consider a function with $f(x_0)>0, f'(x_0)<0, f''(x_0)>0$. This says that near $x_0$ the graph of the function is above the x axis, the function is decreasing, and the function is concave up (see the graph below left). We can also graph a situation where the function is positive, increasing and concave down (graph below right). Notice that the tangent line at the three points is always positive (the function is increasing and $f'(x_0)>0$) but as we move from left to right, the slope gets less positive (or more negative). That means the change in the slope is negative so $f''(x_0)<0$. It also means that the graph is curving downward or is concave down.

Now consider the situation where the graph of the function is the highest of any nearby point at $x_0$ (graph below left). We call such point a local maximum. It's pretty easy to see that at such a point $f(x_0)=0, f''(x_0)<0$ – the function is neither increasing nor decreasing at $x$ and the graph is concave down. Alternatively, consider the situation where the graph of the function is the lowest of any nearby point at $x_0$ (graph below right). There, $f'(x_0)=0, f''(x_0)>0$ the slope of the tangent line is equal to zero and the function is concave up.
The points where a function switches between being concave up and concave down are called inflexion points. At these points, $f''(x_0) = 0$. (They are also the points where the derivative switches from being an increasing to a decreasing function.) So now given a graph of $f(x_0)$, we can make a graph of $f''(x_0)$ and $f'(x_0)$. 