In this section, the problems we address relate to rescaling or shifting the graph of a function. While this is an important topic in itself, the goals of this exercise are much broader. In particular, this provides a great example of translating between descriptions of a function: as a formula, as a graph and as an English description of a procedure. So if you focus on taking shortcuts just to get the answer, you’re selling yourself short. Scaling and shifting functions won’t be so important later in the class, but translating between functions, graphs and English will be. Extra brain pain now will pay off later in the course!

The problems that we’re considering are most easily formulated as figuring out how a given function $F(x)$ relates to another function $G(x)$ that is defined by simple modifications to the function $F$. For example, how is the graph of $G(x) = 3F(x)$ related to the graph of $F(x)$? We’ll first start with a rule based understanding of how this work, and then practice translating things in order to understand the ‘why’ behind the rules.

The Rules:

Suppose you have a the graph of the function $F(x)$. Then

1. The graph of function $G(x) = F(x)+2$ is just the graph of $F$ shifted vertically by 2 units (move up).
2. The graph of function $G(x) = 2F(x)$ is just the graph of $F$ scaled vertically by a factor of 2 (it expands).
3. The graph of the function $G(x) = F(x+2)$ is just the graph of $F$ shifted horizontally by -2 units (move to the left).
4. The graph of function $G(x) = F(2x)$ is just the graph of $F$ scaled horizontally by a factor of $\frac{1}{2}$ (it shrinks).

Exercise: use a graphing calculator to test these rules out using the function $F(x) = x^3/3 - x$.

Of course there is nothing special about the number 2 and one can substitute any number, call it $a$:

1. The graph of function $G(x) = F(x)+a$ is just the graph of $F$ shifted vertically by $a$ units. If $a$ is negative, you shift down.
2. The graph of function $G(x) = aF(x)$ is just the graph of $F$ scaled vertically by a factor of $a$. If $|a|<1$ then the graph “shrinks.” If $a$ is negative, then the graph also “flips” vertically.
3. The graph of the function $G(x) = F(x+a)$ is just the graph of $F$ shifted horizontally by $-a$ units.
4. The graph of function $G(x) = F(ax)$ is just the graph of $F$ scaled horizontally by a factor of $1/a$. If $|a|<1$ then the graph “expands.” If $a$ is negative, then the graph also “flips” horizontally.
Exercise: Test the rules out for several values of using the function $F(x) = x^3/3 - x$. Stop when you are comfortable in seeing the pattern.

**Horizontal vs. Vertical**

You should have noticed that manipulations “on the outside” of the parentheses affect the vertical dimension. Manipulation “inside” the parentheses affect the horizontal dimension and also are “backward” or “opposite” of what you might think. Why?

The first answer has to do with a “function machine” picture of what is going on.

**Rule 1:** $G(x) = F(x) + 2$

**Rule 2:** $G(x) = 2F(x)$

**Rule 3:** $G(x) = F(x + 2)$

**Rule 4:** $G(x) = F(2x)$

For Rule 1 and 2 we manipulate this height by adding 2 or multiplying by 2. In the function machine, the function “add 2” or “multiply by 2” affect the output of $F$, which is plotted vertically. In rule 3 and 4 the function “add 2” or “multiply by 2”
affects the input to F. Since the input to F is put in on the horizontal axis, these manipulations affect things horizontally.

Why are rules 3 and 4 “reverse.” It’s a bit harder to answer, but the natural point to start in understanding the graph is the input to the function F. In rules 1 and 2 we just need to follow things in the forward direction: apply F and then apply “add 2” or “multiply by 2” afterward. But to for rules 3 and 4 the input to the function F is specified ‘in the middle’ of the function box G. From there we apply F in the forward direction to get to the output, but we need to go backward through “add 2” or “multiply by 2” to get to the input to the function G. This going backward is where we get the “reverse” action on the horizontal axis.

**Guess and then check; try it in English**

OK, so the function machine explanation wasn’t the greatest. To get a better understanding, we’ll practice translating the function back and forth into English, or at least a ‘mathese’ form of English that is somewhere between math and English.

Let’s start part way, and suppose you’ve learned the basic properties of the rules. It’s easy to remember the rule that multiplication leads to scaling while adding leads to shifting. It’s also pretty easy to remember/understand why doing things on the outside of the parentheses (changing the output of the function) leads to vertical changes while doing things on the inside of the parentheses (changing the input to the function) leads to horizontal manipulations. But it’s not always easy to remember if the vertical or horizontal direction is the one that is backward.

Instead of trying to reason out whether the vertical or horizontal if flipped, let’s use a perfectly OK method of doing mathematics: guessing. OK so just guessing is not really OK, but guessing and checking IS!
To get the ball rolling, let’s try rule 1 (an easy one). Suppose the solid line shown below is the graph of \( F(x) \) and we’re told to come up with the graph of \( G(x) = F(x) + 2 \). We know that the graph of \( G \) is just the graph of \( F \) shifted by 2 units, but suppose what we don’t know whether it should be shifted up or reversed and shifted down.

So let’s try both possibilities and see which one checks out. The dashed lines are just the graph of \( F \) shifted up and down by 2. To check out which of these possibilities is the real \( G \), we just plug in a few values for \( x \) and see whether the equation \( G(x) = F(x) + 2 \) is true for those values.

For example, let’s find \( G(1) \). \( G(1) = F(1) + 2 \). From the graph \( F(1) = -1 \) so \( F(1) + 2 = 1 \), so the graph of \( G \) is the upper graph.

That works OK, but the real reason that we are doing this exercise is to work on our math to English/mathese translation skills. So how do we translate ‘\( F(x) \)’? We read by saying ‘\( F \) of \( x \)’, but what does that mean. In terms of a procedure, “\( F(x) \) is equal to the value you get when you input the value \( x \) into the function \( F \).” What about in terms of a graph? “\( F(x) \) is the height of the graph of \( F \) above/below the point \( x \) on the horizontal axis.” So how do we translate the equation \( G(x) = F(x) + 2 \) into a mathese sentence describing graphs? Here’s one try: “To find the height of the graph of \( G \) above/below the point \( x \) you find the height of the graph of \( F \) above/below the point \( x \), and then add 2.” From there it’s pretty easy so see that the graph of \( G \) is just the graph of \( F \) increased (raised up) by 2.
Now let’s do the same thing for rule 2, \( G(x) = 2F(x) \). We can guess that the graph of \( G \) is just the graph of \( F \) scaled by 2 (stretched) or by \( \frac{1}{2} \) (compressed) in the vertical direction.

So we can draw both of them and then find \( G(1) \). \( G(1) = F(1) + 2 \). From the graph \( F(1) = -1 \) so \( 2F(1) = -2 \), so the graph of \( G \) is the graph that has been stretched.

In English, \( G(x) = 2F(x) \) can be translated as “to find the height of the graph of \( G \) above/below the point \( x \) you find the height of the graph of \( F \) above/below the point \( x \), and then multiply by 2.” From there it’s pretty easy so see that the graph of \( G \) is just the graph of \( F \) stretched by a factor of 2.
Rules 1 and 2 are the easy ones. How about rule 3? (We’ll use $G(x) = F(x+1)$ to keep the graphs better.) Suppose we know that the graph of $G$ is just the graph of $F$ shifted either left or right by 1 unit. So we can draw both of them.

Let’s find $G(1)$. $G(1) = F(1+1)=F(2)$. From the graph $F(2)=4$ so $G(1)=4$, so the real graph of $G$ is the graph that has been shifted to the left. To understand this backwards business, let’s try English: “to find the height of the graph of $G$ above the point $x$ you find the height of the graph of $F$ above the point $x+1$.” Since the point $x+1$ is to the right of $x$, the graph of $F$ should be to the right of the graph of $G$. Of course if the graph of $F$ is to the right of the graph of $G$, then the graph of $G$ is to the left of the graph of $F$. The last bit is where the reverse comes from and it only really begins to make sense once you translate to English.
Now let’s try rule 4. \( G(x) = F(2x) \). We know that the graph of \( G \) is just the graph of \( F \) scaled horizontally by 2 or by \( 1/2 \). So we can draw both of them.

Let’s find \( G(1) \). \( G(1) = F(2*1) = F(2) \). From the graph \( F(2) = 4 \) so \( G(1) = 4 \). We can also find \( G(-1) \). \( G(-1) = F(2*(-1)) = F(-2) = -4 \). So the graph of \( G \) is the graph that has been **compressed** (the inner graph). In English: “to find the height of the graph of \( G \) above/below the point \( x \) you find the height of the graph of \( F \) above/below the point 2 times \( x \).” Since the horizontal point \( 2x \) is twice as far from the vertical axis, the graph of \( F \) “is further out” than the graph of \( G \). But if the graph of \( F \) is further out than the graph of \( G \), the graph of \( G \) is “compressed inward” relative to the graph of \( F \).

Translating a problem back and forth between “English” the graph and the formula is a critical skill. Practice this problem until you get that “aha” feeling and it starts to make sense. Sometimes you just have to go over and over the same argument. In particular, you need to read the rather awkward English sentence many times until you “feel” how the backward part in rules 3 and 4 work out. You can use the same graphs plotted above to practice what happens when you subtract/divide rather than add/multiply.