MAT1193 – 5a Derivative rules for linear and quadratic functions

Drawing secant lines, taking slopes, and then taking limits of these slopes for each value of the base point is not a practical way for finding the derivative. Luckily there are rules that tell us how to find the derivatives for each of our basic functions. There are also rules to find the derivative of functions built from our rules for combining functions. By applying these rules in the right combination, we can find the derivative for almost any reasonable function we can write down, no matter how complicated.

We will consider three basic perspectives for studying rules for taking the derivative.

1. The derivation of the rule. A derivation is a formal argument for how the rule follows from the definition of the derivative as the limit of the slopes of secant lines.

2. An intuition for the rule. This is a less formal examination of ‘why’ behind the rule to see whether it ‘makes sense.’ Getting a good feel for the rule really helps to understand when the rule would apply in real world situations and what the rule says about how your system works.

3. The application of the rule. If you believe someone else’s derivation and don’t worry about the intuition, you can memorize the rules and learn to apply them to calculate the derivative for complicated functions.

Not every rule has an intuitively understandable explanation, so we’ll only do step 2 for a few of our rules. Also, we won’t go over the derivation for every rule, but it’s important to get an idea of what such a derivation looks like. That’s the main goal of this lecture.

To start with an example, suppose that \( f(x) = x^2 \). Let’s use the definition of the derivative to find the slope of the tangent line above the base point \( x_0 = 2 \). We want to consider the slope of the secant lines passing through our base point and a point on the graph near the base point. So let’s pick at x value near \( x_0 \) call the new x value \( x_1 \). Then the distance from the base point to the new x value is \( \Delta x = x_1 - x_0 \). It’s actually more standard to pick \( \Delta x \) and then let \( x_1 = x_0 + \Delta x \). So let’s find the slope of the secant line through the points \((x_0, f(x_0))\) and \((x_1, f(x_1))\).

\[
\Delta f = \frac{(x_1)^2 - (x_0)^2}{\Delta x} = \frac{4x_0 + 2*\Delta x + \Delta x^2 - 4x_0}{\Delta x} = \frac{4\Delta x + \Delta x^2}{\Delta x} = 4 + \Delta x
\]

Now we want to take the limit of the slopes of the secant line as the point \( x_1 \) gets closer and closer to \( x_0 \). So we want

\[
\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} (4 + \Delta x) = 4
\]
So we the rate of change of the function \( f(x) = x^2 \) with respect to \( x \) near the base point \((2,4)\) is equal to 4. This means that if we pick a number close to \( x_0 = 2 \), we can use a simple linear approximation to get the value of the function. For example if we want to find the approximate value of \((2.1)^2\) we think of 2.1 as \( x_0 + \Delta x \) with \( x_0 = 2 \) and \( \Delta x = 0.1 \). Then since \( \Delta x \) is pretty small, we can approximate the rate of change as

\[
\frac{\Delta f}{\Delta x} = 4 \quad \Rightarrow \quad \Delta f = 4 \Delta x = 0.4
\]

Since \( f(x_0) = 4 \) and \( \Delta f = 0.4 \) we guess that \( f(2.1) = (2.1)^2 = 4.4 \). If you put it in a calculator you find \((2.1)^2 = 4.41\), so our approximation is pretty close.

Now let’s do the derivation for any base point \( x_0 \). We calculate the slope of the secant lines exactly as before

\[
slope = \frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}
\]

\[
\frac{\Delta f}{\Delta x} = \frac{(x_0 + \Delta x)^2 - (x_0)^2}{\Delta x} = \frac{2x_0 \Delta x + \Delta x^2 - (x_0)^2}{\Delta x} = \frac{2x_0 \Delta x}{\Delta x} + \Delta x^2 = 2x_0 + \Delta x
\]

Taking the limit as \( \Delta x \) gets small

\[
\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0
\]

So that gives us a \textit{rule} for finding the derivative of the function \( f(x) = x^2 \) for \textit{any} base point. So if I want to find the slope of the tangent line through the base point \((3,9)\), I use the above formula to find that the slope is \( 2 \times 3 = 6 \). If the base point is \((-1,1)\), then the slope of the tangent line there is \( 2 \times (-1) = -2 \). Here’s a graph showing the original function in blue and the three tangent lines corresponding to \( x = -1, 2, \) and 3.
The rule for taking a base point in, and spitting out the slopes of tangent line is a procedure, and that procedure is known at the derivative of the original function. So putting our rule into the form of a function and using the prime notation for the derivative, we've derived the fact that for the function \( f(x) = x^2, f'(x) = 2x \).

Let's use the derivative to find an approximate value for \((-3.05)^2\). We can think of -3.05 as the base point \( x_0 = -3 \) and a small deviation \( \Delta x = -0.05 \). Near the base point \( x_0 = -3, \) \( \frac{\Delta f}{\Delta x} \approx 2 \cdot (-3) = -6 \rightarrow \Delta f \approx -6\Delta x \)

So combining \( \Delta f = -6 \cdot -0.05 = 0.3 \) with \( f(x_0) = (-3)^2 = 9 \), we find that \((-3.05)^2\) is approximately equal to \( 9 + 0.3 = 9.3 \). Finding the exact number with the calculator gives \((-3.05)^2 = 9.3025 \) so we're pretty close.

Finally, suppose we want to a rule to find the derivative of \( h(z) = 4z^2 \) or \( p(q) = -7q^2 \)? Both of these have the same form as the function \( f(x) = a^x + b \) where \( a \) is a parameter. (\( a = 4 \) for the function \( h(z) = 4z^2 \) and \( a = -7 \) for the function \( p(q) = -7q^2 \)). So let's find the derivative as a limit of secant lines near the base point \( x_0 \).

\[
\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{a(x_0 + \Delta x)^2 - a(x_0)^2}{\Delta x} = a\left(\frac{x_0}{\Delta x}\right)^2 + 2x_0\frac{\Delta x}{\Delta x} + a\Delta x^2 - a\left(\frac{x_0}{\Delta x}\right)^2 = a\frac{2ax_0\Delta x}{\Delta x} + a\Delta x^2 = 2ax_0 + a\Delta x
\]

Taking the limit as \( \Delta x \) gets small,

\[
\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} (2ax_0 + a\Delta x) = 2ax_0
\]

since if \( \Delta x \) goes to zero, then \( a\Delta x \) also goes to zero. Now that we derived our rule, we can use it to find the derivative of \( h(z) = 4z^2 : h'(z) = 8z \). Similarly for \( p(q) = -7q^2 \): \( p'(q) = -14q \).

We can also use the definition to find the formula for the derivative of a line - it's pretty easy. So let's take the slope intercept formula for a general line, \( f(x) = mx + b \) where \( m \) is the slope and \( b \) in y intercept. Letting \( (x_0, f(x_0)) \) be our base point, let's find the slope of the secant line through the points \( (x_0, f(x_0)) \) and \( (x_0 + \Delta x, f(x_0 + \Delta x)) \).

\[
slope = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\left(\frac{m(x_0 + \Delta x) + b}{\Delta x}\right) - \left(\frac{mx_0 + b}{\Delta x}\right)}{\Delta x} = \frac{mx_0 + m\Delta x + b - mx_0 - b}{\Delta x} = \frac{m\Delta x}{\Delta x} = m
\]

This derivation says that the slope of any secant line is always equal to \( m \), no matter what \( \Delta x \) is or no matter what the base point is. This is intuitively obvious from a graph since if we take two points on the graph of a line and draw the secant
line between those points, we’ll get exactly the same graph as the line. So the slope of the secant line is always just the slope of the original line. So we can make a derivative **rule** for linear functions: if $f(x)=m*x+b$, the derivative $f'(x)$ is just the constant function $f'(x) = m$. 