MAT1193 – 6b Derivatives and oscillatory solutions to discrete time dynamical systems

You probably didn’t notice but in all the examples related to the derivative and stability so far, the derivative of the updating function was positive or zero. What happens when the derivative is negative? If we go back to the dynamical system for the distance from the equilibrium, \( \Delta s_{t+1} = h'(s^*) \Delta s_t \), we see that the updating function will multiply by a negative number. We can see what this means by following a chain of arguments:

- If \( s_t \) is a little bigger than \( s^* \), then \( \Delta s_0 \) is positive
- If \( \Delta s_t \) is positive and \( h'(s^*) \) is negative, then \( \Delta s_{t+1} = h'(s^*) \Delta s_t \) is negative
- If \( \Delta s_{t+1} \) is negative, then \( s_{t+1} = s^* + \Delta s_{t+1} \) is smaller than \( s^* \)

A similar argument shows that if if \( s_t \) is a little smaller than \( s^* \), then \( s_{t+1} \) is bigger than \( s^* \). So if \( h'(s^*) < 0 \), solutions near the equilibrium \( s^* \) oscillate back and forth across the value \( s^* \).

What about stability? Stability depends on whether solutions are getting closer to or are moving away from the equilibrium not on whether \( \Delta s_t \) is positive or negative. So for an equilibrium to be stable we want

\[
\frac{|\Delta s_{t+1}|}{|\Delta s_0|} = |h'(s^*)| < 1
\]

If

\[
\frac{|\Delta s_{t+1}|}{|\Delta s_0|} = |h'(s^*)| > 1
\]

Then the solutions are growing in size and the equilibrium is unstable.

So now we have four basic cases for what happens to trajectories near an equilibrium \( s^* \):

- \( h'(s^*) > 1 \) – equilibrium is unstable and trajectories are non-oscillatory. Solutions have a smooth exponential growth.
- \( 0 < h'(s^*) < 1 \) – equilibrium is stable and trajectories are non-oscillatory. Solutions have a smooth exponential decay.
- \( -1 < h'(s^*) < 0 \) – equilibrium is stable and trajectories are oscillatory. Solutions oscillate back and forth, with the magnitude showing an exponential decay. Solutions ‘spiral in’ toward \( s^* \).
• $h'(s^*)<-1$ – equilibrium is **unstable** and trajectories are **oscillatory**. Solutions oscillate back and forth, with the magnitude showing exponential growth. Solutions ‘spiral out’ away from $s^*$.

Here are some examples in terms of graphs. The graph on the left shows the updating function and the result of cobwebbing 10 time steps. The graph on the right shows the solution function. In all cases there is an equilibrium at $s=0.5$.

$h'(s^*) = 1.2$, $s_0 = 5.1$

$h'(s^*) = .8$, $s_0 = 5.75$
\[ h'(s') = -0.8, \ s_0 = 5.75 \]

\[ h'(s') = -1.2, \ s_0 = 5.1 \]